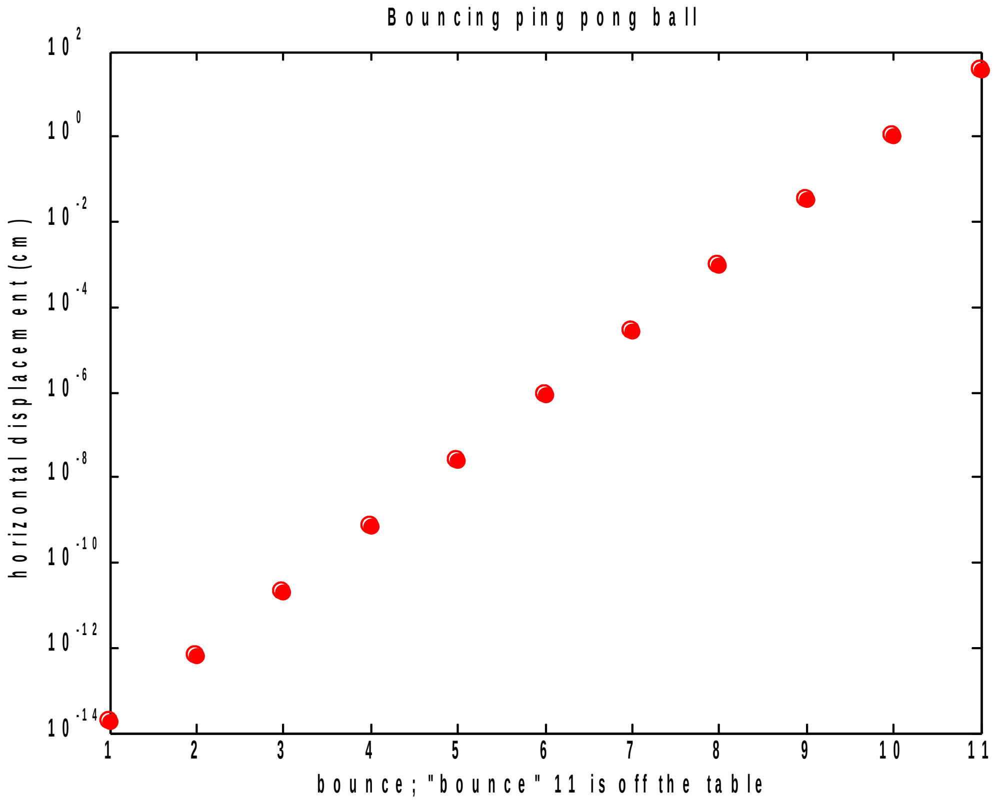
QUANTUM MECHANICS OF A BOUNCING PING PONG BALL

Exploration of Quantum Effects In a Bouncing Ping Pong Ball

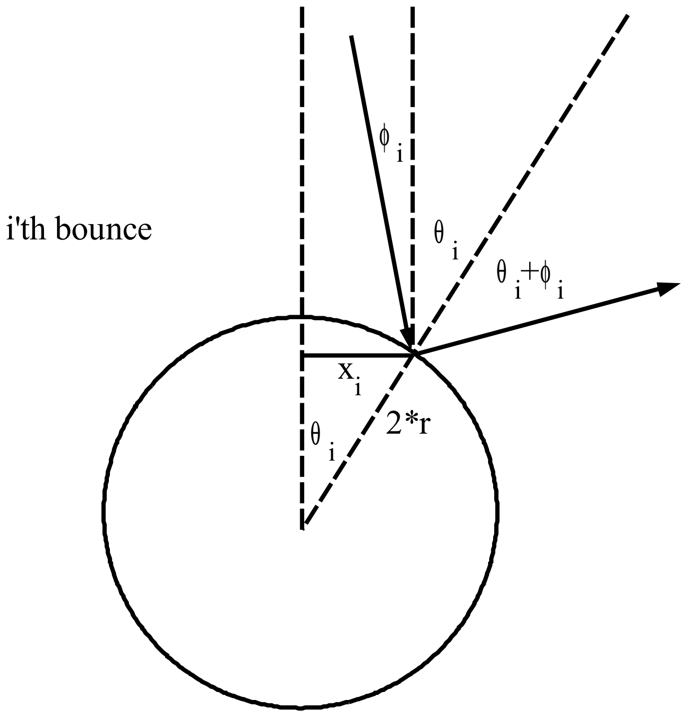
A ping pong ball, with its center separated from the center of another ping pong ball by 10 radii, and dropped onto the other ping pong ball, will make a maximum of 10 bounces, if the collisions are elastic. (The location of the first bounce is determined by the Heisenberg uncertainty relations.)



Below are given, for each bounce, the horizontal displacement of the center of the bouncing ball, as measured from the vertical line drawn through the center of the bottom ball, as well as the increase in horizontal velocity, and the angle between the line joining the centers of the two balls and the vertical axis:

Bounce	Displacement ($\mathbf{x}_i$ centimeters)	Horizontal Velocity ($\mathbf{v}_{xi}$ cm/sec)	Angle from vertical ($\boldsymbol{\theta}_i$ degrees)
1	1.7822e-014	1.5869e-012	2.5672e-013
2	5.9134e-013	5.3946e-011	8.4703e-012
3	2.0088e-011	1.8326e-009	2.8773e-010
4	6.8239e-010	6.2254e-008	9.7745e-009
5	2.3181e-008	2.1148e-006	3.3204e-007
6	7.8748e-007	7.1841e-005	1.1280e-005
7	2.6751e-005	0.0024405	0.00038318
8	0.00090875	0.082904	0.013017
9	0.030871	2.8162	0.44219
10	1.0487	92.022	15.199
(11)	34.306	92.022	90

Here is a figure showing the details for the i'th bounce:



φ_i = collision angle with respect to the vertical, for the i'th bounce
$\varphi_i + \theta_i$ = angle of incidence = angle of reflection, for the i'th bounce
$\varphi_i = 2*\theta_{i-1} + \varphi_{i-1}$
...
$\varphi_2 = 2*\theta_1 + \varphi_1$
$\varphi_1 = \varepsilon$ = epsilon = initial collision angle
x_i = horizontal displacement, for the i'th bounce

Here is the Matlab code which generates these results:

```
% Bouncing ping pong ball; three dimensional analysis
clear
clc
m = 2.4;
r = 2.0;
g = 980;
hbar = 1.0546e-27;
% s = initial separation between the surfaces of the two balls
% s should be expressed in centimeters
% IF: Distance between centers is 10 radii;
% THEN: Distance, s, between surfaces, is 8 radii;
% AND THEN: Ball falls a distance of only 8 radii.
s = 8*r;
t0 = sqrt(2*s/g);                    % t0 = time to fall distance s
v0 = g*t0;                           % maximum vertical speed
x1 = 2*sqrt(hbar*t0/m);              % location of first bounce
% epsilon = initial collision angle with respect to the vertical axis
epsilon = sqrt(hbar/m*t0)/v0;
x(1) = x1;
theta(1) = asin(x(1)/(2*r));
theta(1) = theta(1) + epsilon/2;
% Angle of incidence = angle of reflection, for each bounce.
```

```
vx(1) = v0 * sin(2*theta(1));
i=1;
while x(i) < 2*r
i=i+1;
x(i) = x(i-1) + vx(i-1)*2*t0;            % Displacement
theta(i) = asin(x(i)/(2*r));              % Angle
vx(i) = v0 * sin(2*(sum(theta)));      % Horizontal Velocity
end
N=i-1;
vx(i)=vx(N);
theta = rad2deg(theta);
fprintf('\n\n        Number of bounces = %g\n\n',N)
disp([blanks(5) 'x(cm)' blanks(6) 'vx(cm/sec)'  blanks(3)...
'theta(deg)'])
disp([x' vx' theta'])
semilogy(1:N+1,x,'or','MarkerFaceColor','r')
fname = ['bounce; "bounce" ' num2str(N+1) ' is off the table'];
xlabel(fname)
ylabel('horizontal displacement (cm)')
title('Bouncing ping pong ball')
```

Errors will begin to appear in the results for the final bounces if the initial separation, s, between the surfaces of the balls, is significantly less than one radius, owing to the approximations made in these calculations that the maximum speed, v0, and the time between collisions, 2*t0, are the same for each bounce.

Radius of ball = r = 2.0 cm
Mass of ball = m = 2.4 grams
Acceleration of gravity = g = 980 cm/sec^2
Planck's constant/2*pi = hbar = 1.0546e-27 erg-sec

The mass of the ball enters into the calculations only through the momenta p_x and p_y in the Heisenberg uncertainty relations: $\Delta x \Delta p_x$ = hbar/2, $\Delta y \Delta p_y$ = hbar/2. (Momentum = mass multiplied by velocity.)

The location of the first bounce, x1, is obtained by minimizing:

$\Delta x + \Delta p_x t_0/m = \Delta x + \text{hbar}*t_0/2m\Delta x$

by differentiating with respect to Δx, setting the result equal to zero, and solving for Δx_{min}; doing the same to find Δy_{min}; and then setting:

$x1 = \text{sqrt}(\Delta x_{min}^2 + \Delta y_{min}^2) + \text{sqrt}(\Delta p_{x\,min}^2 + \Delta p_{y\,min}^2)*t0/m$,

where the minimum uncertainty relations are used:

$\Delta x_{min} \Delta p_{x\,min}$ = hbar/2 and $\Delta y_{min} \Delta p_{y\,min}$ = hbar/2.

The initial collision angle is:

$\tan\boldsymbol{\varepsilon} \sim \boldsymbol{\varepsilon} =$ epsilon = $\text{sqrt}(\Delta p_{x\,min}^2 + \Delta p_{y\,min}^2)/mv_0$.

These formulas yield:

$\Delta x_{min} = \Delta y_{min}$ = sqrt(hbar*t0/2*m)

x1 = 2*sqrt(hbar*t0/m)

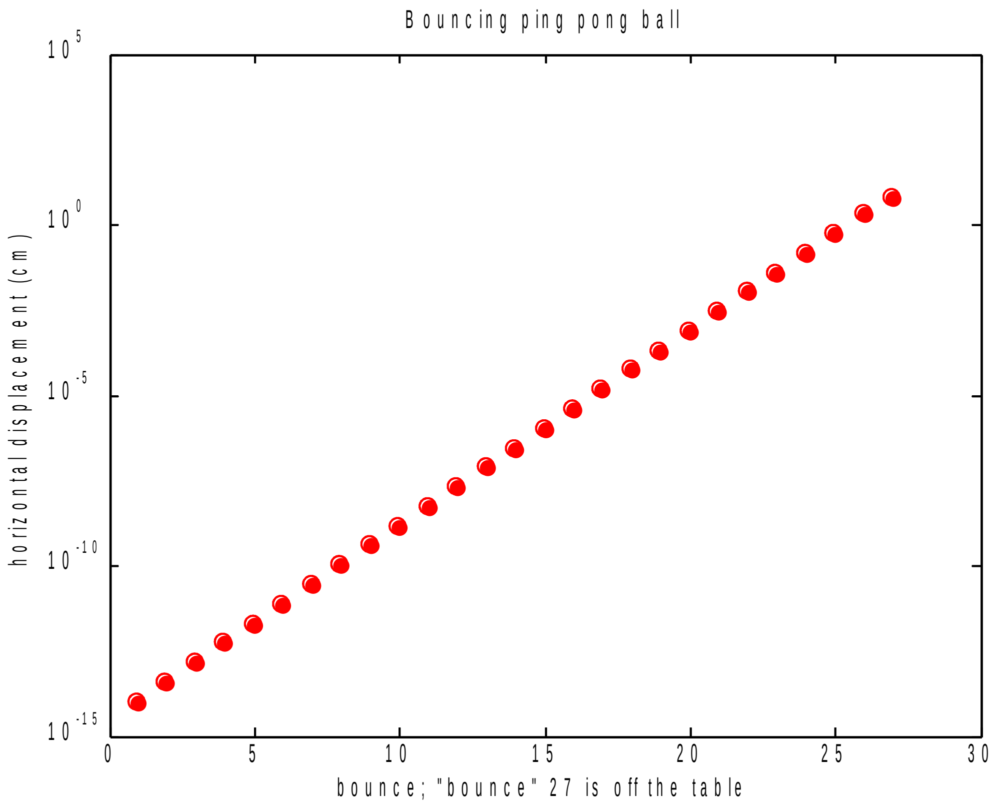
epsilon = sqrt(hbar/m*t0)/v0

When s = 8*r = 16 cm, as in the above example, x1 = 1.78e-014 cm. Compare to the classical electron radius: e^2/mc^2 = 2.82e-013 cm. The minimum distance of the first bounce from zero displacement under ideal conditions is less than 1/10 th the size of a classical electron radius!

IT CAN BE NOTED HERE THAT IF THE INITIAL SEPARATION BETWEEN THE CENTERS OF THE BALLS IS 12 RADII (SO THAT THE INITIAL DISTANCE, s, BETWEEN THE SURFACES OF THE BALLS, IS 10 RADII), THEN THE MAXIMUM NUMBER OF BOUNCES IS 9. THE HORIZONTAL DISPLACEMENT OF THE 9 'TH BOUNCE IS 0.178 CM.

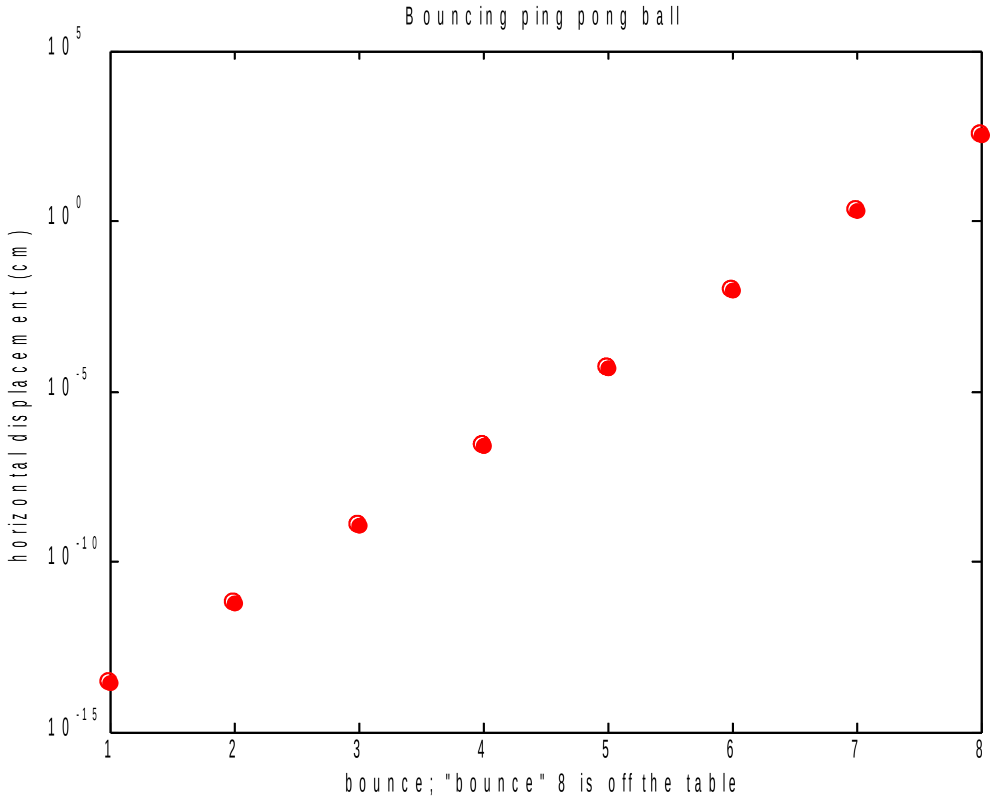
Here are some more examples:

If the initial distance, s, between the surfaces of the balls, is one cm, then the maximum number of bounces is 26:



Bounce	Displacement ($\mathbf{x}_i$ centimeters)	Horizontal Velocity ($\mathbf{v}_{xi}$ cm/sec)	Angle from vertical ($\boldsymbol{\theta}_i$ degrees)
1	8.9109e-015	2.0171e-013	1.3052e-013
2	2.7135e-014	8.0237e-013	3.8868e-013
3	9.963e-014	3.0078e-012	1.4271e-012
4	3.7138e-013	1.1229e-011	5.3197e-012
5	1.3859e-012	4.1907e-011	1.9852e-011
6	5.1722e-012	1.564e-010	7.4087e-011
7	1.9303e-011	5.8369e-010	2.765e-010
8	7.204e-011	2.1784e-009	1.0319e-009
9	2.6886e-010	8.1297e-009	3.8511e-009
10	1.0034e-009	3.0341e-008	1.4372e-008
11	3.7447e-009	1.1323e-007	5.3639e-008
12	1.3975e-008	4.2259e-007	2.0018e-007
13	5.2157e-008	1.5771e-006	7.4709e-007
14	1.9465e-007	5.8859e-006	2.7882e-006
15	7.2645e-007	2.1967e-005	1.0406e-005
16	2.7111e-006	8.198e-005	3.8834e-005
17	1.0118e-005	0.00030595	0.00014493
18	3.7761e-005	0.0011418	0.00054089
19	0.00014093	0.0042614	0.0020186
20	0.00052595	0.015904	0.0075337
21	0.0019629	0.059354	0.028116
22	0.0073255	0.22151	0.10493
23	0.027339	0.82665	0.39161
24	0.10203	3.0829	1.4616
25	0.38057	11.393	5.4595
26	1.4099	36.785	20.639
(27)	4.7335	36.785	90

If the initial distance, s, between the surfaces of the balls, is 100 cm, then the maximum number of bounces is 7:



Bounce	Displacement ($\mathbf{x}_i$ centimeters)	Horizontal Velocity ($\mathbf{v}_{xi}$ cm/sec)	Angle from vertical ($\boldsymbol{\theta}_i$ degrees)
1	2.8179e-014	6.2517e-012	4.0454e-013
2	5.6766e-012	1.2628e-009	8.1312e-011
3	1.1467e-009	2.5508e-007	1.6425e-008
4	2.3162e-007	5.1526e-005	3.3177e-006
5	4.6786e-005	0.010408	0.00067015
6	0.0094505	2.1024	0.13537
7	1.9089	372.48	28.505
(8)	338.45	372.48	90

ANALYTIC SOLUTION

Using the small argument expansions for angles, and keeping only the leading power in the expansion parameter:

$$2*v0*t0/r = 4*s/r,$$

and using the formulas:

$$x(i) = x(i-1) + vx(i-1)*2*t0$$
$$theta(i) = asin(x(i)/(2*r))$$
$$vx(i) = v0 * sin(2*(sum(theta)))$$

with

$$theta(1) = asin(x(1)/(2*r)) + \varepsilon/2,$$

the following expression for the position of the N 'th bounce is obtained:

$$x_N = (4*s/r)^{N-1}[x_1 + r*\varepsilon].$$

SPECIAL CASE: Initial separation of the centers of the balls is 10 radii (the surfaces are separated by 8 radii): then

$$s = 8*r$$
$$4*s/r = 32$$

To find N, let:

$$x_N = 2*r$$

Then

$$2*r = 32^{N-1}[\ x_1 + r*\varepsilon]$$

$$1 = 32^{N-1}[\ x_1/2r + \varepsilon/2]$$

$$0 = (N-1)\log(32) + \log[\ x_1/2r + \varepsilon/2]$$

When r = 2 cm and s = 8*r,

$$x_1 = 1.78 * 10^{-14} \text{ cm}$$
$$\varepsilon = 5.03 * 10^{-17}$$

and

$$N-1 = -\log[\ x_1/2r + \varepsilon/2]/\log(32)$$

$$N = 1 - \log(4.48*10^{-15)}/1.505$$

$$N = 1 - \log(4.48)/1.505 + 15/1.505$$

$$N = 1 - 0.43 + 10 = 11 - 0.43$$

$$N = 10.57$$

Since N is less than 11 when $x_N = 2*r$, and since N must be an integer, the total number of bounces must be 10. This result agrees with the previously determined value for N.

To find the position of the last bounce, which should be less than 2*r, the formula

$$x_N = (4*s/r)^{N-1}[x_1 + r*\varepsilon]$$

is used again, this time with N = 10. With or without using logs, it is found from

$$x_{10} = 32^9 * 1.79 * 10^{-14} \text{ cm}$$

that

$$x_{10} = 0.628 \text{ cm},$$

which is somewhat less than the previously computed value:

$$x_{10} = 1.05 \text{ cm}.$$

ADDITIONAL CONSIDERATIONS

The experiment should be done in vacuum at low temperatures to avoid air currents and thermal effects (vibrations). Thermal effects are expected to always dominate quantum effects in an experiment like this.

Factors reducing the number of bounces include the Coriolis effect, rotation or spin of the bouncing ball, imperfections in the balls, human error, etc.

Inelasticity (energy loss per bounce) actually serves to increase the number of bounces by reducing the increase in horizontal velocity gained with each bounce.